A Classical Monetary Model

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Lecture II
Basic Facts

- Perfect competition (homogenous good)
- Fully flexible prices in all markets
- Very limited role to money - serves as a unit of account
  - Later we introduce money to utility function
- Introduces notation and assumptions on preferences and technology
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A representative household seeks to maximize objective function

\[ E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) \]  

Utility function has common properties

Subject to a flow of budget constraints

\[ P_t C_t + Q_t B_t = B_{t-1} + W_t N_t - T_t \]  

\( B_t \) represents the quantity of one-period, nominally riskless discount bonds purchased in period \( t \) and maturing in period \( t + 1 \)

Household is assumed to take as given the price of the good, the wage and the price of bonds

And is subject to a solvency constraint

\[ \lim_{t \to \infty} E_t \{ B_t \} \geq 0 \]
Households

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P_t C_t + Q_t B_t = B_{t-1} + W_t N_t - T_t \tag{2}
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- \(B_t\) represents the quantity of one-period, nominally riskless discount bonds purchased in period \(t\) and maturing in period \(t + 1\)
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\[
\lim_{t \to \infty} E_t \{B_t\} \geq 0 \tag{3}
\]
Optimal Consumption and Labor Supply

Optimality conditions are given by

\[- \frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t} \tag{4}\]

\[Q_t = \beta E_t \left\{ \frac{U_{c,t+1}}{U_{c,t}} \frac{P_t}{P_{t+1}} \right\} \tag{5}\]

Assuming that the period utility function takes the form

\[U(C_t, N_t) = C_t^{1-\sigma} - N_t^{1+\varphi} \tag{6}\]

Optimality conditions become

\[C_t^\sigma N_t^\varphi = \frac{W_t}{P_t} \tag{7}\]

\[Q_t = \beta E_t \left\{ \left( \frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right\} \tag{8}\]
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Firms

- A representative firm uses production function (technology)
  \[ Y_t = A_t N_t^{1-\alpha} \]  

Each period the firm maximizes profit

\[ P_t Y_t - W_t N_t \]  

Subject to production function and taking the price and wage as given

Profit maximization results in optimality condition

\[ \frac{W_t}{P_t} = (1 - \alpha) A_t N_t^{-\alpha} \]  

Firm hires labor up to the point where its marginal product equals the real wage

Equivalently, the marginal cost must be equated to the price \( P_t \)

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Steady State and the Balanced Growth Path

- We assume constant growth of consumption and constant inflation in the steady state

\[
\frac{\bar{C}_{t+1}^{-\sigma}}{\bar{C}_t^{-\sigma}} = \gamma \\
\frac{\bar{P}_{t+1}}{\bar{P}_t} = \bar{\Pi}
\]

(13)  (14)

- Consumption growth (and the real wage) is driven by the exogenous growth in technology \(A\)

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- Steady-state level of consumption and the real wage are not constant and follow a balanced growth path
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- Steady-state level of consumption and the real wage are not constant and follow a balanced growth path
Steady State Solution

Households

\[ \bar{C}^\sigma \bar{N}^\varphi = \frac{\bar{W}}{\bar{P}} \]  \hspace{1cm} (16)

\[ \bar{Q} = \beta \left\{ \left( \frac{\bar{C} \gamma}{\bar{C}} \right)^{-\sigma} \frac{\bar{P}}{\bar{P} \bar{\Pi}} \right\} \]  \hspace{1cm} (17)

\[ \bar{Q} = \beta \gamma^{-\sigma} \frac{1}{\bar{\Pi}} \]  \hspace{1cm} (18)

Firms

\[ \bar{Y} = \bar{A} \bar{N}^{1-\alpha} \]  \hspace{1cm} (19)

\[ \frac{\bar{W}}{\bar{P}} = (1 - \alpha) \bar{A} \bar{N}^{-\alpha} \]  \hspace{1cm} (20)
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\frac{\bar{W}}{\bar{P}} = (1 - \alpha)\bar{A}\bar{N}^{-\alpha} \tag{20}
\]
Steady State Solution

- Steady state level of nominal and real interest rates is given by

\[
\bar{Q} = \beta \gamma^{-\sigma} \frac{1}{\Pi} \tag{21}
\]

\[
\bar{R} = \beta \gamma^{-\sigma} \tag{22}
\]

- And after taking log

\[
\bar{i} = \rho + \sigma \gamma - \bar{\pi} \tag{23}
\]

\[
\bar{r} = \rho + \sigma \gamma \tag{24}
\]
Steady State Solution

- Steady state level of labor and production comes out combining (16), (19) and (20) and using $C = Y$

\[
\bar{N} = \nu_{na} \bar{A}^{\psi_{na}} \tag{25}
\]
\[
\bar{Y} = \nu_{ya} \bar{A}^{\psi_{ya}} \tag{26}
\]

- Where

\[
\psi_{na} = \frac{1 - \sigma}{\sigma(1 - \alpha) + \phi + \alpha} \tag{27}
\]
\[
\psi_{ya} = \frac{1 + \phi}{\sigma(1 - \alpha) + \phi + \alpha} \tag{28}
\]
\[
\nu_{na} = (1 - \alpha) \frac{1}{\sigma(1 - \alpha) + \phi + \alpha} \tag{29}
\]
\[
\nu_{ya} = \nu_{na}^{1-\alpha} \tag{30}
\]
Log-linear approximation around the steady-state

- Definitions according to Uhlig (1999)

\[
\tilde{x}_t = \ln X_t - \ln \bar{X} 
\]  
\[
X_t = \bar{X} e^{\tilde{x}_t} 
\]

- His rules are

\[
e^{\tilde{x}_t} + a\tilde{y}_t \approx 1 + \tilde{x}_t + a\tilde{y}_t 
\]
\[
\tilde{x}_t\tilde{y}_t \approx 0 
\]
\[
E_t \left[ ae^{\tilde{x}_{t+1}} \right] \approx a + aE_t \left[ \tilde{x}_{t+1} \right] 
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Log-linear approximation around the steady-state

- Euler equation

\[
Q_t = \beta E_t \left\{ \left( \frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right\}
\]

\[
\bar{Q} e^{q_t} = \beta E_t \left\{ \left( \frac{\bar{C}^{-\sigma} \gamma^{-\sigma} e^{-\sigma c_{t+1}}}{\bar{C}^{-\sigma} e^{-\sigma c_t}} \right) \frac{\bar{P} e^{p_t}}{\bar{P} \bar{\Pi} e^{p_{t+1}}} \right\}
\]

\[
\bar{Q} = \beta \gamma^{-\sigma} \frac{1}{\bar{\Pi}}
\]

\[
e^{q_t} = \left( \frac{e^{-\sigma E_t\{c_{t+1}\}}}{e^{-\sigma c_t}} \right) \frac{e^{p_t}}{e E_t\{p_{t+1}\}}
\]

\[
e^{q_t} = e^{-\sigma E_t\{c_{t+1}\}} + \sigma c_t + p_t - E_t\{p_{t+1}\}
\]

\[
1 + q_t = 1 - \sigma E_t\{c_{t+1}\} + \sigma c_t + p_t - E_t\{p_{t+1}\}
\]

\[
-i_t = -\sigma E_t\{c_{t+1}\} + \sigma c_t + p_t - E_t\{p_{t+1}\}
\]

\[
c_t = E_t\{c_{t+1}\} - \frac{1}{\sigma} (i_t - (E_t\{p_{t+1}\} - p_t))
\]
Log-linear approximation of optimality conditions

- **Households**

  \[ c_t = E_t\{c_{t+1}\} - \frac{1}{\sigma} (i_t - E_t\{\pi_{t+1}\}) \]  \hfill (36)

  \[ w_t - p_t = \sigma c_t + \varphi n_t \]  \hfill (37)

- **Firms**

  \[ w_t - p_t = a_t - \alpha n_t + \ln (1 - \alpha) \]  \hfill (38)

  \[ y_t = a_t + (1 - \alpha)n_t \]  \hfill (39)

- **And for a moment somehow arbitrarily**

  \[ m_t - p_t = y_t - \eta i_t \]  \hfill (40)
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Equilibrium

- In principle the Equilibrium is similar to the steady-state solution
- We start assuming $c_t = y_t$
- Then using the above and combining optimality conditions with the log-linear aggregate production function

$$n_t = \psi_{na} a_t + \nu_{na}$$ (41)
$$y_t = \psi_{ya} a_t + \nu_{ya}$$ (42)

- Where

$$\psi_{na} = \frac{1 - \sigma}{\sigma(1 - \alpha) + \phi + \alpha}$$ (43)
$$\psi_{ya} = \frac{1 + \phi}{\sigma(1 - \alpha) + \phi + \alpha}$$ (44)
$$\vartheta_{na} = \frac{\log(1 - \alpha)}{\sigma(1 - \alpha) + \phi + \alpha}$$ (45)
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In principle the Equilibrium is similar to the steady-state solution. We start assuming $c_t = y_t$. Then using the above and combining optimality conditions with the log-linear aggregate production function:

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Furthermore, given the equilibrium process for output, Euler equation can be used to determine the implied real interest rate, $r_t \equiv i_t - E_t i_{t+1}$, as

$$ r_t = \rho + \sigma E_t \{ \Delta y_{t+1} \} $$

(47)

$$ = \rho + \sigma E_t \{ \Delta a_{t+1} \} $$

(48)

Finally, the equilibrium real wage $\omega_t \equiv w_t - p_t$ is given by

$$ \omega_t = a_t - \alpha n_t + \log(1 - \alpha) $$

(49)

Note that the equilibrium dynamics of employment, output, and the real interest rate are determined independently of monetary policy.

All real variables fluctuate in response to variations in technology.
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Note that the equilibrium dynamics of employment, output, and the real interest rate are determined *independently of monetary policy*. All real variables fluctuate in response to variations in technology.
Equilibrium

- **Output and the real wage always rise in the face of the productivity increase**
- The sign of employment is ambiguous depending on whether $\sigma$ is larger or smaller than one
  - When $\sigma < 1$, the substitution effect on labor supply resulting from a higher real wage dominates the negative effect caused by a smaller marginal utility of consumption
  - The opposite is true whenever $\sigma > 1$
  - And employment remains unchanged with $\sigma = 1$, i.e. when the utility of consumption is logarithmic
- **Response of the real interest rate critically depends on the time series properties of technology**
  - If the current improvement in technology is transitory, $E_t\{a_{t+1}\} < a_t$, then the real rate will go down
  - If the technology is expected to keep improving, $E_t\{a_{t+1}\} > a_t$, the real rate will increase with the rise in $a_t$
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  - The opposite is true whenever \( \sigma > 1 \).
  - And employment remains unchanged with \( \sigma = 1 \), i.e. when the utility of consumption is logarithmic.
- Response of the real interest rate critically depends on the time series properties of technology.
  - If the current improvement in technology is transitory, \( E_t\{a_{t+1}\} < a_t \), then the real rate will go down.
  - If the technology is expected to keep improving, \( E_t\{a_{t+1}\} > a_t \), the real rate will increase with the rise in \( a_t \).
Not surprisingly, equilibrium behavior of nominal variables (inflation, nominal interest rate, nominal wage) cannot be determined uniquely by real forces.

It requires specification of how monetary policy is conducted.

Before we move to that recall the Fisher equation

\[ i_t = r_t - E_t\{\pi_{t+1}\} \]  

That implies that the nominal rate adjusts one for one with expected inflation given the real interest rate that is determined solely by real factors.
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An Exogenous Path for the Nominal Interest Rate

Consider first the case of the nominal interest rate following an exogenous stationary process $i_t$ and assume that $i_t$ has a mean $\rho$ consistent with a steady state with zero inflation and no growth.

Using Fisher equation, write

$$E_t\{\pi_{t+1}\} = i_t - r_t$$

(51)

where $r_t$ is determined independently of monetary policy.

While previous equation pins down expected inflation there is no other condition determining current inflation. It follows that any path for the price level satisfying

$$p_{t+1} = p_t + i_t - r_t + \xi_t$$

(52)

where $\xi_t$ is a shock, possibly unrelated to economic fundamentals, satisfying $E_t\{\xi_{t+1}\} = 0$ for all $t$.

Such shocks are referred as sunspot shocks and the example above shows how an exogenous nominal interest rate leads to price level indeterminacy.
Consider first the case of the nominal interest rate following an exogenous stationary process $i$ and assume that $i_t$ has a mean $\rho$ consistent with a steady state with zero inflation and no growth.

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Such shocks are referred as sunspot shocks and the example above shows how an exogenous nominal interest rate leads to price level indeterminacy.
Simple Inflation-Based Interest Rate Rule

- Suppose now that the central bank adjusts the nominal interest rate according to the rule

\[
i_t = \rho + \Phi_\pi \pi_t
\]  
(53)

where \( \Phi_\pi > 0 \)

- Combining this rule with Fisher equation yields

\[
\Phi_\pi = E_t\{\pi_{t+1}\} + \hat{r}_t
\]  
(54)

where \( \hat{r}_t \equiv \rho - r_t \)

- If \( \Phi_\pi > 1 \), the previous difference equation has only one stationary solution (solution that remains in the vicinity of the steady state)

- Solution can be obtained by solving (54) forward

\[
\pi = \sum_{k=0}^{\infty} \Phi_{\pi}^{-(k+1)} E_t\{\hat{r}_{t+k}\}
\]  
(55)

This equation fully determines inflation as a function of the path of the real interest rate.
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\[ i_t = \rho + \Phi \pi_t \]  

(53)

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\[ \pi = \sum_{k=0}^{\infty} \Phi^{-(k+1)} E_t \{ \hat{r}_{t+k} \} \]  

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This equation fully determines inflation as a function of the path of the real interest rate.
In contrary, if $\Phi_\pi < 1$, the stationary solutions to (54) take the form

$$\pi_{t+1} = \Phi_\pi \pi_t - \hat{r}_t + \xi_{t+1}$$

where \(\{\xi_{t+1}\}\) is, again, an arbitrary sequence of shocks, satisfying $E_t\{\xi_{t+1}\} = 0$ for all $t$.

It follows that any process $\{\pi_t\}$ satisfying previous equation is consistent with equilibrium.

As in the case of exogenous nominal rate the price level (and, hence, inflation) are not determined uniquely.

The condition for a determinate price level, $\Phi_\pi$, is a property known as the *Taylor principle*.
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An Exogenous Path for the Money Supply

- Now suppose that the central bank sets an exogenous path for the money supply $\{m_t\}$
- Eliminating interest rate in Fisher equation using the money demand yields following difference equation for the price level

$$p_t = \left(\frac{\eta}{1 + \eta}\right) E_t\{p_{t+1}\} + \left(\frac{1}{1 + \eta}\right) m_t + u_t$$

where $u_t \equiv (1 + \eta)^{-1}(\eta r_t - y_t)$ evolves independently of $\{m_t\}$
- If $\eta > 0$ the above difference equation can be solved forward as

$$p_t = \frac{1}{1 + \eta} \sum_{k=0}^{\infty} \left(\frac{\eta}{1 + \eta}\right)^k E_t\{m_{t+k}\} + u'_t$$

where $u'_t \equiv \sum_{k=0}^{\infty} \left(\frac{\eta}{1 + \eta}\right)^k E_t\{u_{t+k}\}$ is, again, independent of $\{m_t\}$
Now suppose that the central bank sets an exogenous path for the money supply \( \{m_t\} \).

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p_t = \frac{1}{1 + \eta} \sum_{k=0}^{\infty} \left( \frac{\eta}{1 + \eta} \right)^k E_t\{m_{t+k}\} + u'_t \tag{58}
\]

where \( u'_t \equiv \sum_{k=0}^{\infty} \left( \frac{\eta}{1 + \eta} \right)^k E_t\{u_{t+k}\} \) is, again, independent of \( \{m_t\} \)
Previous forward solution can be rewritten in terms of the expected future growth rate of money

\[ p_t = m_t \frac{1}{1 + \eta} \sum_{k=1}^{\infty} \left( \frac{\eta}{1 + \eta} \right)^k E_t \{ \Delta m_{t+k} \} + u_t' \]  

(59)

Hence, an arbitrary exogenous path for the money supply always determines the price level uniquely.
Finally, given the price level, money demand can be used to solve for the nominal interest rate

\[
i_t = \eta^{-1}[y_t - (m_t - p_t)] \tag{60}
\]

\[
i_t = \eta^{-1} \sum_{k=1}^{\infty} \left( \frac{\eta}{1 + \eta} \right)^k E_t\{\Delta m_{t+k}\} + u'_t \tag{61}
\]

where \(u''_t \equiv \eta^{-1}(u'_t + y_t)\) is independent of monetary policy.
In a basic classical model the households’ utility is a function of consumption and hours only - two variables that are invariant to the way monetary policy is conducted.

That means that there is no policy rule better than any other - monetary policy causing huge fluctuation in nominal variables is no less desirable that one stabilizing prices in the face of the same shock.

Clearly, this result is extreme and empirically unappealing and we will overcome it with versions of the classical model in which a motive to keep part of the households’ wealth in money is introduced explicitly.
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